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Solitons and excitations in the duality-based matrix model

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Abstract: We analyse a specific, duality-based generalization of the hermitean matrix model. The existence of two collective fields enables us to describe specific excitations of the hermitean matrix model. By using these two fields, we construct topologically non-trivial solutions (BPS solitons) of the model. We find the low-energy spectrum of quantum fluctuations around the uniform solution. Furthermore, we construct the wave functional of the ground state and obtain the corresponding Green function.

Keywords: Field Theories in Lower Dimensions, Solitons Monopoles and Instantons, Matrix Models.

Contents

1. Introduction

The recently renewed interest in matrix models came from the identification [1] of matrix quantum mechanics with the worldvolume theory of unstable $D0$-branes in the two-dimensional string theory. One uses matrix models to study non-perturbative aspects of string theory, or more specifically, the open-closed string duality.

In the usual approach, the closed string states are related to the matrix model states by bosonization of the fermion field followed by a non-local field redefinition [2]. Alternatively, one can use the collective-field formulation of the matrix model to obtain a continuous, field-theoretic description of the model. This approach was used to construct the collective-field formulation of unitary [3] and symplectic [4] matrix models. The important result of the collective-field formulation was the analysis of higher-order terms in the $1/N$ expansion. It was shown that divergences cancelled and the finite correction, of order $1/N^2$, was calculated [5].

The two-dimensional string theory is obtained from a matrix model in the double scaling limit, with the critical potential given by the inverted harmonic oscillator. It was shown in Ref.[6] that one could remove the oscillator potential from the action by a suitable coordinate transformation. Therefore, it is of interest to study general features of the background independent theory without the external potential.

Furthermore, it was realized that in order to describe black-hole solution, the generalization of the standard hermitean matrix model is needed [8]. One such generalization was proposed in the paper [9].

In this paper, we analyze this duality-based generalization [9] and interpret the model in question as hermitean matrix model beyond the usual singlet approximation. We construct non-trivial solitonic solutions of the model, and find the spectrum of quantum fluctuations around the uniform density solution of the master Hamiltonian constructed in Refs.[7, 9].

The master Hamiltonian can be interpreted as a two-family Calogero model, connected by duality, with mutually inverse coupling constants [7], or, for a special choice of coupling strength ($\lambda = 1/2$), as a master Hamiltonian for symmetric and quaternionic models [9]. In Ref.[9] it was conjectured that the master Hamiltonian with $\lambda = 1/2$ actually corresponded to the hermitean matrix model, and our analysis of the model confirms this conjecture. From the string point of view, we describe oriented string degrees of freedom (hermitean matrix model) by unoriented string degrees of freedom (symmetric and quaternionic matrix model)[1]. Although we focus on the matrix model with $\lambda = 1/2$, constructed in Ref.[9], throughout the paper we write λ for generality. Also, in this way, the $\lambda \leftrightarrow 1/\lambda$ duality is obvious. The aim of this investigation is to shed more light on the nature of duality in the field-theoretic description of matrix models.

2. Semiclassical solutions of the master Hamiltonian

Let us start with the master Hamiltonian obtained in Ref.[9]:

$$
\begin{aligned}
H^M = & \frac{1}{2}\int dx\rho(x)(\partial_x\pi_\rho(x))^2 + \frac{1}{8}\int dx\rho(x)\left[(\lambda-1)\frac{\partial_x\rho(x)}{\rho(x)} + 2\lambda \fint dy\frac{\rho(y)}{x-y} + 2\fint dy\frac{m(y)}{x-y}\right]^2 + \\
& + \frac{\lambda}{2}\int dx m(x)(\partial_x\pi_m(x))^2 + \frac{\lambda}{8}\int dx m(x)\left[(\frac{1}{\lambda}-1)\frac{\partial_x m(x)}{m(x)} + \frac{2}{\lambda}\fint dy\frac{m(y)}{x-y} + 2\fint dy\frac{\rho(y)}{x-y}\right]^2 - \\
& - \frac{\lambda}{2}\int dx\ \rho(x)\partial_x\frac{P}{x-y}\bigg|_{y=x} - \frac{1}{2}\int dx\ m(x)\partial_x\frac{P}{x-y}\bigg|_{y=x}.
\end{aligned} \tag{2.1}
$$

The collective fields $\rho(x)$ and $m(x)$ are normalized as

$$\int dx\rho(x) = N,\ \int dx m(x) = M, \tag{2.2}$$

where N and M are large numbers of independent matrix eigenvalues (or particles in the system for the Calogero model), and $\pi_{\rho/m}(x)$ represent the canonical momenta:

$$[\partial_x\pi_\rho(x), \rho(y)] = -i\partial_x\delta(x-y),\ [\partial_x\pi_m(x), m(y)] = -i\partial_x\delta(x-y). \tag{2.3}$$

One can add a term $-\mu_\rho \int dx\rho(x)$ and an analogous one for $m(x)$ to the Hamiltonian to ensure (2.2).

The leading part[2] of the collective-field Hamiltonian in the $1/N$ and $1/M$ expansions is given by the positive-definite effective potential, the second and the fourth term in the Hamiltonian (2.1). Owing to the positive definiteness of the effective potential, the Bogomol'nyi limit appears. The Bogomol'nyi bound is saturated by the positive normalizable

[1]One can think of our master Hamiltonian as an explicit realization of an idea proposed in [10], that one can continuously deform oriented string theories in two dimensions into unoriented ones by turning on non-local interactions on the worldsheet.

[2]Note that the divergent terms (the last line in the Hamiltonian (2.1), as well as the momenta, are suppressed in the leading order, but are contributing to the spectrum of quantum fluctuations.

solutions $\rho_0(x)$ and $m_0(x)$ of the coupled equations

$$(\lambda-1)\frac{\partial_x\rho(x)}{\rho(x)}+2\lambda\fint dy\frac{\rho(y)}{x-y}+2\fint dy\frac{m(y)}{x-y}=0, \tag{2.4}$$

$$\left(\frac{1}{\lambda}-1\right)\frac{\partial_x m(x)}{m(x)}+\frac{2}{\lambda}\fint dy\frac{m(y)}{x-y}+2\fint dy\frac{\rho(y)}{x-y}=0. \tag{2.5}$$

An obvious solution is a constant density for both fields:

$$\rho_0=N/L,\ \text{and}\ m_0=M/L. \tag{2.6}$$

This solution exists on the compact support only, as can be seen from the normalization conditions (2.2).

Next, let us look for non-trivial solutions of the coupled equations (2.4) and (2.5). Substracting these equations and then integrating them, we find the condition

$$\rho(x)m(x)=c, \tag{2.7}$$

where c is an arbitrary constant parameter. Expressing $m(x)$ and introducing it in eq. (2.4), we are left with

$$(\lambda-1)\partial_x\rho(x)+2\lambda\rho(x)\fint\frac{dy\rho(y)}{x-y}+2c\rho(x)\fint\frac{dy}{\rho(y)(x-y)}=0. \tag{2.8}$$

Let us take the following ansatz

$$\rho(x)=\rho_0\frac{x^2+a^2}{x^2+b^2}, \tag{2.9}$$

where a and b are arbitrary positive constants. Using the Hilbert transform

$$\fint\frac{dy}{x-y}\frac{1}{y-z}=i\pi\frac{sign(Im\ z)}{x-z}, \tag{2.10}$$

we find the conditions

$$\frac{\rho_0\pi}{b}(b^2-a^2)=\frac{1-\lambda}{\lambda},\ \ c=\frac{\lambda\rho_0^2a}{b}. \tag{2.11}$$

The soliton-antisoliton solution is given by

$$\rho(x)=\rho_0+\frac{\lambda-1}{\lambda\pi}\frac{b}{x^2+b^2}, \tag{2.12}$$

$$m(x)=\frac{c}{\rho_0}+\frac{1-\lambda}{\pi}\frac{a}{x^2+a^2}. \tag{2.13}$$

Depending on the sign of $(\lambda-1)$, one density describes a hole in the condensate, whereas the other describes a particle above the condensate. We see that $\lambda\leftrightarrow 1/\lambda$ interchanges holes and particles.

Now let us study the case $c=ka$ in the limit $a\to 0$. In this limit, $m(x)$ goes to the delta function:

$$\lim_{a\to 0}m(x)=\frac{kb^2}{\rho_0}\pi\delta(x) \tag{2.14}$$

and $\rho(x)$ reduces to

$$\lim_{a\to 0}\rho(x) = \rho_0 \frac{x^2}{x^2+b^2}. \tag{2.15}$$

Although these two expressions are obviously solutions of the Bogomol'nyi equations, it is instructive to insert them into eqs. (2.4), (2.5) to confirm the correctness of the limiting procedure. Inserting (2.14) and (2.15) into eq. (2.4), we obtain the conditions

$$k = \frac{\lambda\rho_0^2}{b} \ , \ \pi\rho_0 b = \frac{1-\lambda}{\lambda}. \tag{2.16}$$

Next, we check the validity of eq. (2.5):

$$(1-\lambda)\partial_x m(x) + 2m(x) \fint dy \frac{m(y)}{x-y} + 2\lambda m(x) \fint dy \frac{\rho(y)}{x-y} = 0. \tag{2.17}$$

Inserting (2.14) and (2.15) into (2.17) we find

$$(1-\lambda)\lim_{a\to 0}\frac{ka}{\rho_0}\frac{2x(x^2+a^2)-2x(x^2+b^2)}{(x^2+a^2)^2} + 2\lambda\lim_{a\to 0}\frac{ka}{\rho_0}\frac{x^2+b^2}{x^2+a^2}\lim_{a\to 0}\frac{\pi\rho_0(a^2-b^2)}{b}\frac{x}{x^2+b^2} +$$
$$+2\lim_{a\to 0}\frac{ka}{\rho_0}\frac{x^2+b^2}{x^2+a^2}\lim_{a\to 0}\frac{\pi k(b^2-a^2)}{\rho_0}\frac{x}{x^2+a^2} = 0. \tag{2.18}$$

Using the identity

$$\lim_{b\to 0}\frac{2bx}{(x^2+b^2)^2} = 2\pi\frac{P}{x}\delta(x) = -\pi\partial_x\delta(x) \tag{2.19}$$

and (2.14), we obtain only one condition

$$\frac{\pi k b^2}{\rho_0} = 1-\lambda \tag{2.20}$$

which is obviously consistent with (2.16). This is due to $m(x) \sim \delta(x)$ and (2.17).

Next, we construct the two-soliton solution of eqs. (2.4), (2.5). With the following ansatz for $\rho(x)$ and $m(x)$

$$\begin{aligned} \rho(x) &= \rho_0\frac{(x-a)(x-a^*)(x+a)(x+a^*)}{(x-b)(x-b^*)(x+b)(x+b^*)}, \\ m(x) &= \frac{c}{\rho} = m_0\frac{(x-b)(x-b^*)(x+b)(x+b^*)}{(x-a)(x-a^*)(x+a)(x+a^*)}, \end{aligned} \tag{2.21}$$

we easily find the condition $b = a$, so that the solutions reduce to trivial ones with constant ρ_0 and m_0.

Now, let us take $c = kIm(a)$ in the limit $Im(a) \to 0$, and write the ansatz (2.21) in the more natural form

$$\begin{aligned} \rho(x) &= \lim_{\epsilon\to 0}\rho_0\frac{[(x-x_0)^2+\epsilon^2][(x+x_0)^2+\epsilon^2]}{(x^2-b^2)(x^2-b^{*2})} = \rho_0\frac{(x-x_0)^2(x+x_0)^2}{(x^2-b^2)(x^2-b^{*2})} = \\ &= \rho_0\left(1+\frac{B}{x-b}+\frac{B^*}{x-b^*}-\frac{B}{x+b}-\frac{B^*}{x+b^*}\right) \quad , \end{aligned} \tag{2.22}$$

$$m(x) = \lim_{\epsilon\to 0}\frac{k\epsilon}{\rho_0}\frac{(x^2-b^2)(x^2-b^{*2})}{[(x-x_0)^2+\epsilon^2][(x+x_0)^2+\epsilon^2]} = \tilde{k}\left[\delta(x-x_0)+\delta(x+x_0)\right], \tag{2.23}$$

where

$$
\begin{aligned}
B &= \frac{(b^2 - x_0^2)^2}{2b(b^2 - b^{*2})}, \\
\tilde{k} &= \frac{k\pi(x_0^2 - b^2)(x_0^2 - b^{*2})}{4\rho_0 x_0^2}.
\end{aligned} \tag{2.24}
$$

Inserting the ansatz (2.22) into eq. (2.4) we find the following three conditions:

$$
B = -B^* \ , \ 2\lambda\rho_0\pi i B = \lambda - 1 \ , \ \tilde{k} = 1 - \lambda. \tag{2.25}
$$

After we introduce $b = |b|e^{i\phi}$ and $r = x_0^2/|b|^2 > 0$, the condition $B = -B^*$ gives

$$
r = 2\cos\phi - 1 \Rightarrow x_0 = |b|\sqrt{(2\cos\phi - 1)}, \tag{2.26}
$$

Next, from $\rho_0\pi i B = \lambda - 1$ we find

$$
\rho_0\pi|b| = \frac{1-\lambda}{\lambda}\frac{\cot\frac{\phi}{2}}{2\cos\phi}. \tag{2.27}
$$

So, from (2.25) and (2.26) we have obtained $|b|$ and x_0 as functions of ϕ and the last condition gives k

$$
\frac{2k\pi}{\rho_0}\left[\frac{\cos^2\phi(1-\cos\phi)}{2\cos\phi - 1}\right] = 1 - \lambda, \tag{2.28}
$$

as a function of ϕ.

In the BPS limit, one can construct n-soliton solutions, $n > 2$, following the strategy demostrated above for the two-soliton solution. The general ansatz for n-soliton solution is of the form

$$
\begin{aligned}
m(z) &= \sum_{i=1}^{n} m_0\delta(z - x_i), \\
\rho(x) &= \rho_0\left(1 + \sum_{\alpha=1}^{n}\left(\frac{B_\alpha}{x - z_\alpha} + \frac{B_\alpha^*}{x - z_\alpha^*}\right)\right) \\
&= \rho_0\frac{\prod_{i=1}^{n}(x - x_i)^2}{\prod_{\alpha=1}^{n}(x - z_\alpha)(x - z_\alpha^*)}.
\end{aligned} \tag{2.29}
$$

Note that this ansatz satisfies the condition $\rho(x)m(x) = 0$. To obtain relations between poles and zeros explicitly, one has to solve the general algebraic problem of finding zeros of the polynomial of degree $n \geq 3$. The soliton solutions have the following properties: they are located at the point at which the $\rho(x)$ field is vanishing, but the $m(x)$ field is becoming undetermined. This property is also characteristic of BPS solutions describing monopols and Julia-Zee dyons [11].

3. Quantum excitations

3.1 The spectrum of quantum fluctuations

At this point we analyse the dynamics of the collective-field excitations around the ground-state solution (2.6) of our master Hamiltonian (2.1). First, we rewrite the master Hamiltonian in the following form:

$$H^M = \frac{1}{2}\int dx\rho(x)A_0^\dagger(x)A_0(x) + \frac{\lambda}{2}\int dx m(x)B_0^\dagger(x)B_0(x), \tag{3.1}$$

where

$$\begin{aligned} A_0(x) &= \partial_x\pi_\rho(x) + i\left[\frac{(\lambda-1)}{2}\frac{\partial_x\rho(x)}{\rho(x)} + \lambda\fint dy\frac{\rho(y)}{x-y} + \fint dy\frac{m(y)}{x-y}\right], \\ B_0(x) &= \partial_x\pi_m(x) + i\left[\frac{(1-\lambda)}{2\lambda}\frac{\partial_x m(x)}{m(x)} + \frac{1}{\lambda}\fint dy\frac{m(y)}{x-y} + \fint dy\frac{\rho(y)}{x-y}\right]. \end{aligned} \tag{3.2}$$

Next, we perform the $1/N$ $(1/M)$ expansion of the collective field $\rho(x)$ $(m(x))$

$$\rho(x) = \rho_0 + \partial_x\eta(x),\ m(x) = m_0 + \partial_x\tilde{\eta}(x), \tag{3.3}$$

where $\partial_x\eta(x)$ and $\partial_x\tilde{\eta}(x)$ are small density quantum fluctations[3]. We insert (3.3) into the Hamiltonian (3.1) and expand up to second-order terms in $\partial_x\eta(x)$ and $\partial_x\tilde{\eta}(x)$. We obtain a Hamiltonian quadratic in fluctuations:

$$H^{(2)} = \frac{\rho_0}{2}\int dx A^\dagger A + \frac{\lambda m_0}{2}\int dx B^\dagger B. \tag{3.4}$$

The operators $A(x)$ and $B(x)$

$$A(x) = -\pi_\eta(x) + i\left[\frac{(\lambda-1)}{2}\frac{\partial_x^2\eta}{\rho_0} + \fint\frac{dy}{x-y}(\lambda\partial_y\eta(y) + \partial_y\tilde{\eta}(y))\right], \tag{3.5}$$

$$B(x) = -\pi_{\tilde{\eta}}(x) + i\left[\frac{(1-\lambda)}{2\lambda}\frac{\partial_x^2\tilde{\eta}}{m_0} + \fint\frac{dy}{x-y}(\partial_y\eta(y) + \frac{1}{\lambda}\partial_y\tilde{\eta}(y))\right], \tag{3.6}$$

satisfy the following commutation relations:

$$\left[A(x), A^\dagger(y)\right] = -\frac{\lambda-1}{\rho_0}\partial_x\partial_y\delta(x-y) + 2\lambda\partial_x\frac{P}{x-y}, \tag{3.7}$$

$$\left[B(x), B^\dagger(y)\right] = -\frac{1-\lambda}{\lambda m_0}\partial_x\partial_y\delta(x-y) + \frac{2}{\lambda}\partial_x\frac{P}{x-y}, \tag{3.8}$$

$$\left[A(x), B^\dagger(y)\right] = 2\partial_x\frac{P}{x-y}. \tag{3.9}$$

Now, to find the spectrum of low-lying excitations, we have to diagonalize the Hamiltonian (3.4). We expand the operators $A(x)$ and $B(x)$ in terms of new, complete sets of

[3]The small density quantum fluctations are defined by the explicit derivative because of the normalization condition $\int dx\partial_x\eta(x) = 0$.

operators

$$A(x) = \sum_n \phi_n(x) a_n, \; A^\dagger(x) = \sum_n \phi_n^*(x) a_n^\dagger, \tag{3.10}$$

$$B(x) = \sum_n \varphi_n(x) b_n, \; B^\dagger(x) = \sum_n \varphi_n^*(x) b_n^\dagger, \tag{3.11}$$

where

$$\left[a_n, a_m^\dagger\right] = \omega_n \delta_{nm}, \; \left[b_n, b_m^\dagger\right] = \Omega_n \delta_{nm}, \; \left[a_n, b_m^\dagger\right] \neq 0. \tag{3.12}$$

We demand that the Hamiltonian (3.4) should take the following form:

$$H^{(2)} = \sum_n a_n^\dagger a_n + \sum_n b_n^\dagger b_n. \tag{3.13}$$

Generally, n is the quantum number, and it is assumed that the sum is replaced by an integral for a continuous spectrum.

Let us concentrate on the first part of the Hamiltonian (3.13). We insert the expansion (3.10) in the commutators (3.7) and apply (3.12) to obtain the completeness relation

$$\sum_n \omega_n \phi_n(x) \phi_n^*(y) = -\frac{\lambda - 1}{\rho_0} \partial_x \partial_y \delta(x-y) + 2\lambda \partial_x \frac{P}{x-y}. \tag{3.14}$$

Inserting (3.10) in (3.4) and demanding (3.13), we obtain an orthogonality relation

$$\frac{\rho_0}{2} \int dx \phi_n^*(x) \phi_m(x) = \delta_{nm}. \tag{3.15}$$

Next, we multiply the relation (3.15) by $\phi_m^*(y)$, sum over m, apply the completeness relation (3.14) and finally obtain the equation for the functions $\phi_n(x)$:

$$\omega_n \phi_n(x) = \frac{\lambda - 1}{2} \partial_x^2 \phi_n(x) + \lambda \rho_0 \partial_x \fint dy \frac{\phi_n(y)}{x-y}. \tag{3.16}$$

The plane waves are the solution of Eq.(3.16) with

$$\omega_n = \frac{1-\lambda}{2} n^2 + \lambda \rho_0 \pi |n|. \tag{3.17}$$

An analogous procedure for the second part of the Hamiltonian (3.13) gives us Ω_n

$$\Omega_n = \frac{\lambda - 1}{2} n^2 + m_0 \pi |n|. \tag{3.18}$$

A few comments are in order here. The relation (3.17) was obtained in Ref.[12] as a dispersion relation for quantum fluctuations around the ground-state solution in the Calogero model. Under the duality transformation $\lambda \to 1/\lambda$ and $\rho_0 \leftrightarrow m_0$, the relations (3.17) and (3.18) are interchanged, reflecting the known duality symmetry $\lambda \to 1/\lambda$ of the Calogero model. However, note that these dispersion relations do not describe physical excitations since $[a, b^\dagger] \neq 0$, i.e., the Hamiltonian (3.13) is not diagonal!

Furthermore, for $\lambda = 1/2$, the relations (3.17) and (3.18) were obtained in Ref.[13]. In this paper, these dispersions represented quantum fluctuations for two independent, effective Hamiltonians operating on two particular subsets of eigenstates of the hermitean matrix model[4]. However, our duality-based generalization of the hermitean matrix model also includes the interaction between these two sets of mutually dual degrees of freedom.

Let us continue with our diagonalization procedure. From the completeness and orthogonality relations we obtain

$$\phi_n(x) = \frac{\exp(inx)}{\sqrt{\pi\rho_0}},\ \varphi_n(x) = \frac{\exp(inx)}{\sqrt{\pi\lambda m_0}}. \tag{3.19}$$

Now, we are in a position to calculate the commutator $\left[a_n, b_m^\dagger\right]$. From (3.9) and (3.15), using (3.17), (3.18), and (3.19), we obtain

$$\left[a_n, b_m^\dagger\right] = \delta_{nm}|n|\pi\sqrt{\rho_0 m_0 \lambda} \equiv \delta_{nm} f_n. \tag{3.20}$$

In the limit of small n, an interesting possibility arises - $a_n \propto b_n$. Appropriate rescaling of the relations (3.12) gives us the following Hamiltonian:

$$H^{(2)} = \sum_n (\omega_n + \Omega_n) c_n^\dagger c_n = \sum_n |n|\pi(\lambda\rho_0 + m_0) c_n^\dagger c_n, \tag{3.21}$$

where the bosonic operators c_n satisfy

$$[c_n, c_m] = [c_n^\dagger, c_m^\dagger] = 0,\ \left[c_n, c_m^\dagger\right] = \delta_{nm} \tag{3.22}$$

We see that in the lowest order, the special case $\lambda = 1/2$ reproduces the part of singlet sector of hermitean matrix model [12], i.e., we can represent the hermitean matrix model (the Calogero model with $\lambda = 1$) as a dual system of symmetric ($\lambda = 1/2$) and quaternionic ($\lambda = 2$) matrices.

Of course, this is to be expected. Although we claim that the collective-field variables used in the construction [9] of the master Hamiltonian (3.1) with $\lambda = 1/2$ do contain some non-singlet states of the hermitean matrix model, these non-singlet states do not contribute in the low-energy approximation, and, therefore, the obtained result is in fact a consistency check. In the string theory language, this means that the unoriented string amplitudes at tree level are the same as in the oriented model.

What can we say about the excitations for "not-so-low" energy? Let us exactly diagonalize the Hamiltonian (3.13). Having the quadratic Hamiltonian allows us to apply the Bogoliubov transformation. We introduce two sets of mutually commuting operators $\tilde{c}_n$ and $\tilde{d}_n$

$$\left[\tilde{c}_n, \tilde{c}_m^\dagger\right] = \omega_n^+ \delta_{nm},\ \left[\tilde{d}_n, \tilde{d}_m^\dagger\right] = \omega_n^- \delta_{nm}, \tag{3.23}$$

[4] These effective Hamiltonians were actually a collective-field formulation of matrix models for the symmetric and quaternionic matrices, compare Ref.[9].

such that we can write the Hamiltonian (3.13) as a sum of two independent quadratic parts:

$$H^{(2)} = \sum_n \tilde{c}_n^\dagger \tilde{c}_n + \sum_n \tilde{d}_n^\dagger \tilde{d}_n. \tag{3.24}$$

Assume that

$$a_n = \alpha_n \tilde{c}_n - \beta_n \tilde{d}_n \text{ and } b_n = \beta_n \tilde{c}_n + \alpha_n \tilde{d}_n, \tag{3.25}$$

where α_n, β_n are some coefficients that depend on the quantum number n. Now, inserting (3.25) into (3.13) and demanding (3.24), and inserting (3.25) into (3.12), gives us a system of four equations

$$\begin{gathered} \alpha_n \beta_n (\omega_n^+ - \omega_n^-) = f_n,\ \alpha_n^2 + \beta_n^2 = 1, \\ \alpha_n^2 \omega_n^+ + \beta_n^2 \omega_n^- = \omega_n,\ \beta_n^2 \omega_n^+ + \alpha_n^2 \omega_n^- = \Omega_n, \end{gathered} \tag{3.26}$$

which we solve. The coefficients in the Bogoliubov transformation (3.25) are

$$\begin{aligned} \alpha_n^2 &= \frac{1}{2}\left(1 + \frac{\omega_n - \Omega_n}{\sqrt{(\omega_n - \Omega_n)^2 + 4f_n^2}}\right), \\ \beta_n^2 &= \frac{1}{2}\left(1 - \frac{\omega_n - \Omega_n}{\sqrt{(\omega_n - \Omega_n)^2 + 4f_n^2}}\right). \end{aligned} \tag{3.27}$$

The spectrum of low-lying excitations is given by

$$\begin{aligned} \omega_n^\pm &= \frac{1}{2}\left[\omega_n + \Omega_n \pm \sqrt{(\omega_n - \Omega_n)^2 + 4f_n^2}\right] = \\ &= \frac{\pi(\lambda\rho_0 + m_0)|n|}{2}\left(1 \pm \sqrt{1 + \frac{2(1-\lambda)(\lambda\rho_0 - m_0)}{\pi(\lambda\rho_0 + m)^2}|n| + \frac{(1-\lambda)^2}{\pi^2(\lambda\rho_0 + m_0)^2}n^2}\right) \end{aligned} \tag{3.28}$$

Note that each branch of the dispersion relation (3.28) is invariant under the duality transformation $\lambda \to 1/\lambda$ and $\rho_0 \leftrightarrow m_0$, and physical excitations are not connected by duality. We have two sets of independent oscillators. Also note that obtained corrections are of the same order as non-singlet corrections [15], namely $1/N^2$ with respect to the leading singlet contribution.

However, for $\lambda = 1/2$ the ground system is dual, i.e., $\lambda\rho_0 = m_0$ and ω_n^- is negative! The "wrong" sign of ω_n^- signalizes the instability of the vacuum.

3.2 The vacuum functional and Green functions

The vacuum functional is a simultaneous solution of the equations

$$A_0(x)|0\rangle = 0 \text{ and } B_0(x)|0\rangle = 0, \tag{3.29}$$

and is given by

$$\begin{aligned} \Psi_0(\rho, m) = \exp\Big\{ & \frac{(\lambda-1)}{2}\int dx \rho(x)\ln\rho(x) + \frac{\lambda}{2}\int\int dxdy\rho(x)\ln|x-y|\rho(y) + \\ & + \int\int dxdy \rho(x)\ln|x-y|m(y) + \\ & + \frac{(1-\lambda)}{2\lambda}\int dx m(x)\ln m(y) + \frac{1}{2\lambda}\int\int dxdy m(x)\ln|x-y|m(y)\Big\}. \end{aligned} \tag{3.30}$$

We expand the vacuum found in (3.30) up to quadratic terms

$$\Psi^{(2)} = \exp\left(\int\int dzdy\, \boldsymbol{\eta}^T(z)\mathbf{G}^{-1}(z-y)\boldsymbol{\eta}(y)\right), \tag{3.31}$$

where we have introduced the matrix notation

$$\begin{aligned}
\boldsymbol{\eta}^T(y) &= \left[\frac{\partial_y \eta(y)}{\sqrt{\rho_0}}\,,\,\frac{\partial_y \tilde{\eta}(y)}{\sqrt{\lambda m_0}}\right] \\
\mathbf{G}^{-1}(z-y) &= -\frac{1}{4\pi}\left[\begin{smallmatrix} (1-\lambda)\pi\delta(z-y)-2\lambda\pi\rho_0\ln|z-y| & -2\pi\sqrt{\lambda\rho_0 m_0}\ln|z-y| \\ -2\pi\sqrt{\lambda\rho_0 m_0}\ln|z-y| & (\lambda-1)\pi\delta(z-y)-2\pi m_0\ln|z-y| \end{smallmatrix}\right] = \\
&= -\frac{1}{4\pi}\int dk \frac{e^{ik(z-y)}}{k^2}\left[\begin{smallmatrix} \omega_k & f_k \\ f_k & \Omega_k \end{smallmatrix}\right]\,.
\end{aligned} \tag{3.32}$$

This vacuum is a solution of the equations

$$A(x)\Psi^{(2)} = 0, B(x)\Psi^{(2)} = 0 \Leftrightarrow \tilde{c}_k\Psi^{(2)} = 0, \tilde{d}_k\Psi^{(2)} = 0 \tag{3.33}$$

Writing $\Psi^{(2)}$ in terms of the field

$$\begin{aligned}
\boldsymbol{\varphi}(x) &= \int dx \mathbf{S}(x-x')\boldsymbol{\eta}(x'), \\
\mathbf{S}(x-x') &= \int dk \frac{e^{ik(x-x')}}{2\pi\sqrt{|k|}(k^2+k_0^2)^{1/4}}\left[\begin{smallmatrix} \alpha_k & \beta_k \\ -\beta_k & \alpha_k \end{smallmatrix}\right]\,,
\end{aligned} \tag{3.34}$$

where $k_0^2 = 4\lambda^2\pi^2\rho_0^2/(1-\lambda)^2$, we obtain

$$\Psi^{(2)} = \exp\left\{\int\int dzdy \boldsymbol{\varphi}^T(z)\left(-\frac{1}{4\pi}\int dk e^{ik(z-y)}\frac{\sqrt{k^2+k_0^2}}{|k|}\left[\begin{smallmatrix} \omega_k^+ & 0 \\ 0 & \omega_k^- \end{smallmatrix}\right]\right)\boldsymbol{\varphi}(y)\right\} \tag{3.35}$$

and the non-normalizability of the vacuum $\Psi^{(2)}$ is manifest. This is a consequence of the commutation relations (3.23) from which we conclude that $\tilde{d}_n$ is actually a creation operator which we call $d_n^\dagger$ and $\tilde{d}_n^\dagger$ is an annihilation operator now called d_n.

The Hamiltonian (3.24) is, after appropriate rescaling, given by

$$H^{(2)} = \sum_n \omega_n^+ c_n^\dagger c_n + \sum_n |\omega_n^-| d_n d_n^\dagger = \sum_n \omega_n^+ c_n^\dagger c_n + \sum_n |\omega_n^-| d_n^\dagger d_n + \sum_n |\omega_n^-|. \tag{3.36}$$

The last term must be included in the vacuum energy. It simply defines the Fermi level of the system.

Solving equations for the normalizable vacuum functional

$$c_k\Phi^{(2)} = 0, d_k\Phi^{(2)} = 0, \tag{3.37}$$

we find

$$\Phi^{(2)} = \exp\left\{\int\int dzdy \boldsymbol{\varphi}^T(z)\mathbf{G}_{\varphi\varphi}^{-1}(z-y)\boldsymbol{\varphi}(y)\right\}. \tag{3.38}$$

The inverse propagator $\mathbf{G}^{-1}_{\varphi\varphi}(z-y)$ is given by

$$\begin{aligned}\mathbf{G}^{-1}_{\varphi\varphi}(z-y) &= -\frac{1}{4}\int dk e^{ik(z-y)}\frac{\sqrt{k^2+k_0^2}}{|k|}\begin{bmatrix}\omega_k^+ & 0\\ 0 & |\omega_k^-|\end{bmatrix} = \\ &= \frac{\lambda-1}{4}(\partial_z\partial_y+k_0^2)\left[\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\delta(z-y)+\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)\frac{k_0}{\pi}K_0(k_0(z-y))\right], \quad (3.39)\end{aligned}$$

where $K_0(x)$ is the Bessel function.

The two components of the field $\boldsymbol{\varphi}(y)$ can be combined into a complex scalar field, which could be interpreted as a complex tachyon field of $0A$ two-dimensional string theory [14].

4. Conclusion

We have analysed the spectrum of quantum fluctuations around a particular ground-state solution of the master Hamiltonian (3.1), representing a duality-based generalization of the hermitean matrix model. This has given us a glimpse of the dynamics of dual degrees of freedom in field theory. We have shown that for $\lambda = 1/2$, the master Hamiltonian describes a new representation of the hermitean matrix model, in which some specific states have been analysed. These states contribute to the next-to-leading order, and the contribution is comparable with the contribution from the non-singlet sector. Also, we are able to construct topologically non-trivial, solitonic solutions of the model in question, which do not exist in the singlet approximation of the hermitean matrix model. In Ref.[15] the non-singlet states were attributed to the vortices on the worldsheet. Note that these non-local interactions on the worldsheet can be interpreted as double-trace deformations of the standard hermitean matrix model originating from the $(\lambda-1)$-proportional terms in the master Hamiltonian [10].

The excitations in our model behave as quasi-particles and quasi-holes, and can be described by a complex scalar field. Therefore, in our opinion, the relation between the matrix model constructed in Ref.[9] and 0A string theory in two dimensions deserves a more detailed analysis. Furthermore, we have constructed the soliton-antisoliton solution for the master Hamiltonian, and quantum fluctuations around this solution are currently analysed [19]. Also, it would be interesting to investigate the possible integrability of the model, as the existence of multisoliton solutions suggests.

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